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1998 J. Phys. A: Math. Gen. 31 3791

(http://iopscience.iop.org/0305-4470/31/16/009)

J. Phys. A: Math. Gen. **31** (1998) 3791–3813. Printed in the UK PII: S0305-4470(98)88589-2

A systematic construction of completely integrable Hamiltonians from coalgebras

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Received 20 October 1997

Abstract. A universal algorithm to construct N-particle (classical and quantum) completely integrable Hamiltonian systems from representations of coalgebras with Casimir elements is presented. In particular, this construction shows that quantum deformations can be interpreted as generating structures for integrable deformations of Hamiltonian systems with coalgebra symmetry. In order to illustrate this general method, the $so(2,1)$ algebra and the oscillator algebra h_4 are used to derive new classical integrable systems including a generalization of Gaudin–Calogero systems and oscillator chains. Quantum deformations are then used to obtain some explicit integrable deformations of the previous long-range interacting systems and a (non-coboundary) deformation of the $(1+1)$ Poincaré algebra is shown to provide a new Ruijsenaars–Schneider-like Hamiltonian.

1. Introduction

It is well known that quantum groups appeared in the context of quantum inverse scattering methods as a new kind of symmetry linked to the integrability of some quantum models constructed in Lax form (see [1–3]). Quantum algebras and groups are related by duality [4] and, in the previous context, the concept of 'quantum algebra invariance' expresses the commutativity of a given Hamiltonian with respect to the generators of a certain quantum algebra. Since their introduction, the construction and analysis of quantum group invariant integrable models has attracted much effort (see [5–9] and references therein) and a great amount of literature has also been devoted to quantum group theory (see, for instance, [10]).

From an abstract mathematical point of view, two ideas were emphasized as a consequence of these developments: the relevance of deformations (in the sense of [11] and [12]) and the concept of Hopf algebra [13]. In particular, quantum algebras are simply defined as Hopf algebra deformations of usual universal enveloping Lie algebras. On the other hand, although quantum semisimple algebras were those initially linked to integrable models, the construction of quantum deformations of non-semisimple Lie algebras has also been successfully explored by using different methods (see [14, 15]).

This paper establishes a general and constructive connection between Hopf algebras and integrability that can be stated as follows. Given *any* coalgebra (A, Δ) with Casimir element C, each of its representations gives rise to a family of completely integrable Hamiltonians $H^{(N)}$ with an arbitrary number N of degrees of freedom. We provide a constructive proof of this statement that contains the explicit definition of such Hamiltonians and their integrals of motion. Moreover, both classical and quantum mechanical systems can be obtained from the same (A, Δ), provided we endow this coalgebra with a suitable additional structure (that will be either a Poisson bracket or a non-commutative product on A, respectively). Note that, instead of Hopf algebra structures, our construction makes use of the more general term of coalgebra (neither the counit nor the antipode mappings will be explicitly used).

It is important to emphasize that the validity of this general procedure by no means depends on the explicit form of Δ (i.e. on whether the coalgebra (A, Δ) is deformed or not). This fact is crucial in order to clarify the significance of quantum algebras (and groups) in our framework: they are 'only' a particular class of coalgebras that can be used to construct systematically integrable systems. However, the specific feature of such systems will be that they are *integrable deformations* of those obtained by the same method when we start from the corresponding non-deformed coalgebra. Moreover, usual Lie algebras are always endowed with a coalgebra structure, and we shall see that many interesting coalgebra-induced systems can be derived from them without making use of any deformation machinery. In this way, a new general application (intrinsically different from the usual ones [16, 17]) of Lie algebras in the field of integrable systems is presented. At this point, we would like to mention that this result concerning non-deformed Lie coalgebras was already proven in [18], and it can also be extracted from [19], but without explicit mention of the underlying coalgebra structure.

In the next section the basics of Hopf algebras are revisited and the definition and properties of Poisson coalgebras presented. Realizations of Poisson coalgebras on canonical coordinates are introduced and coupled with the coproduct map in order to obtain two-particle representations. Section 3 is devoted to the construction of a family of $N = 2$ integrable systems from a $so(2, 1)$ Poisson coalgebra structure. An integrable deformation of this family is afterwards obtained by making use of the (standard) quantum deformation of this algebra. This two-dimensional example contains the seminal ideas of our construction, that need a proper generalization in order to reach the full N-dimensional case. The mathematical improvements needed to succeed in such a general scheme are presented in section 4, that contains an analysis of the usual definition of the Nth coproduct map $\Delta^{(N)} : A \to A \otimes A \otimes \ldots^{N)} \otimes A$ in terms of a recurrence relation that starts with the second-order coproduct $\Delta \equiv \Delta^{(2)}$. It turns out that it is possible to rewrite $\Delta^{(N)}$ in a different way that is much more convenient for our purposes.

Section 5 introduces the general constructive result, valid for both classical and quantum mechanical systems: the Nth coproduct of any (smooth) function of the generators of a coalgebra defines an integrable Hamiltonian whose constants of motion in involution are given by the mth coproducts of the Casimir element C, with $m = 1, \ldots, N$. Functional independence among the constants is guaranteed by construction. Some comments concerning the specific features of both the classical and the quantum mechanical cases are included.

In order to show the direct applicability of these results, section 6 includes various examples based on phase-space realizations of coalgebras (although the quantization of some of them is not difficult, a careful treatment of some quantum mechanical examples will be presented in a forthcoming paper). The first makes use of the classical $so(2, 1)$ Poisson coalgebra in order to construct a new multiparameter generalization of an integrable system that has been recently introduced by Calogero [20] and whose coalgebra symmetry is manifestly extracted. The second non-deformed example is provided by the (primitive) coalgebra linked to the (non-semisimple) oscillator algebra h_4, that leads to a straighforward proof of the integrability of a system of coupled oscillators first given in [21]. Afterwards, the fact that quantum algebras can be interpreted as the generating objects of integrable deformations is illustrated by using the standard quantum deformation of $so(2, 1)$ to obtain—

through suitable Poisson realizations of this quantum algebra—an integrable deformation of the previous $so(2,1)$ family. Finally, another interesting example of a quantum algebra induced integrable system is provided by the Vaksman–Korogodskii deformation [22] of the $(1+1)$ Poincaré algebra, which gives rise to a Ruijsenaars–Schneider-like integrable Hamiltonian [23]. It is interesting to note that, in this case, the system defined through the non-deformed $(1+1)$ Poincaré algebra is quite trivial; consequently, the quantum deformation seems sometimes to be essential in order to produce a dynamically relevant Hamiltonian.

In section 7 a deeper insight into the $so(2,1)$ models is presented by precluding the use of canonical realizations and working with classical 'angular momentum variables'. In this way, the long-range nature of the interaction of these models is clearly appreciated. Under this realization, the non-deformed coalgebra gives rise to the hyperbolic XXX Gaudin magnet [24], and the integrable deformation linked to $U_z(so(2,1))$ is translated into physical terms as the introduction of a variable range exchange [25] within the Gaudin Hamiltonian. Finally, the paper is closed with some remarks concerning open questions and future developments.

2. Coalgebras and Poisson realizations

2.1. Hopf algebras

A *Hopf algebra* is a (unital, associative) algebra $(A,\cdot)$ endowed with two homomorphisms called coproduct $(\Delta : A \longrightarrow A\otimes A)$ and counit $(\epsilon : A \longrightarrow \mathbb{C})$, as well as an antihomomorphism (the antipode $\gamma : A \longrightarrow A$) such that, $\forall a\in A$:

$$(\mathrm{id}\otimes\Delta)\Delta(a) = (\Delta\otimes\mathrm{id})\Delta(a) \tag{2.1}$$

$$(\mathrm{id}\otimes\epsilon)\Delta(a) = (\epsilon\otimes\mathrm{id})\Delta(a) = a \tag{2.2}$$

$$m((\mathrm{id}\otimes\gamma)\Delta(a)) = m((\gamma\otimes\mathrm{id})\Delta(a)) = \epsilon(a)1 \tag{2.3}$$

where m is the usual multiplication mapping $m(a\otimes b) = a\cdot b$. This notion was introduced by Hopf [13] in a cohomological context but, as we shall see, it expresses a basic idea in many-body problems and it is often implicitly used. The aim of this paper is to make its physical significance more explicit, that is basically concentrated within the coproduct Δ. In fact, hereafter we shall deal mainly with coalgebras, i.e. algebras endowed with a coassociative (2.1) coproduct Δ.

For our purposes, the most interesting example of coalgebra is provided by the universal enveloping algebra $U(g)$ of a Lie algebra g with generators X_i. The algebra $U(g)$ can be endowed with a Hopf algebra structure by defining,

$$\begin{aligned}
&\Delta(X_i) = 1\otimes X_i + X_i\otimes 1 \qquad \Delta(1) = 1\otimes 1\\
&\epsilon(X_i) = 0 \qquad \epsilon(1) = 1\\
&\gamma(X_i) = -X_i \qquad \gamma(1) = 1.
\end{aligned} \tag{2.4}$$

These maps acting on the generators of g are straightforwardly extended to any monomial in $U(g)$ by means of the homomorphism condition $\Delta(X\cdot Y) = \Delta(X)\cdot\Delta(Y)$. In general, an element Y of a Hopf algebra such that $\Delta(Y) = 1\otimes Y + Y\otimes 1$ is called *primitive*, and Friedrichs' theorem ensures that, in $U(g)$, the only primitive elements are the generators X_i [26]. On the other hand, the homomorphism condition implies the compatibility of the coproduct Δ with the Lie bracket

$$[\Delta(X_i),\Delta(X_j)]_{A\otimes A} = \Delta([X_i,X_j]_A) \qquad \forall X_i, X_j\in g. \tag{2.5}$$

From a physical point of view, if g is the algebra of observables of some one-particle physical system, the coproduct in (2.4) is just the usual definition of 'total' quantum observables for the two-particle system.

In this context, quantum algebras are just coalgebra deformations of $U(g)$: a deformed, but coassociative, coproduct is defined and a set of (possibly deformed) commutation rules can be found in such a way that the compatibility condition (2.5) is recovered. The whole 'quantum' structure depends on (perhaps more than one) deformation parameters and the non-deformed coalgebra (2.4) is recovered when all the parameters vanish. A well known example is the standard (Drinfel'd–Jimbo [4, 27]) deformation of $U(so(2,1))$ with deformed coproduct

$$\begin{aligned} \Delta(\tilde{J}_2) &= 1 \otimes \tilde{J}_2 + \tilde{J}_2 \otimes 1 \\ \Delta(\tilde{J}_1) &= \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \tilde{J}_1 + \tilde{J}_1 \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \\ \Delta(\tilde{J}_3) &= \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \tilde{J}_3 + \tilde{J}_3 \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \end{aligned} \tag{2.6}$$

and deformed commutation rules compatible with (2.6)

$$[\tilde{J}_2, \tilde{J}_1] = \tilde{J}_3 \qquad [\tilde{J}_2, \tilde{J}_3] = -\tilde{J}_1 \qquad [\tilde{J}_3, \tilde{J}_1] = \frac{\sinh(z\tilde{J}_2)}{z}. \tag{2.7}$$

Another important object is essential for our purposes: the existence of a deformed Casimir that commutes with all the generators of the quantum algebra and, in this case, reads

$$C_z(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3) = \left(2\frac{\sinh(\frac{z}{2}\tilde{J}_2)}{z}\right)^2 - \tilde{J}_1^2 - \tilde{J}_3^2. \tag{2.8}$$

As we shall see, both deformed and non-deformed Casimir elements will be the keystones of the integrability properties of the systems induced from their respective coalgebras.

2.2. Poisson coalgebras and canonical realizations

In general, a *Poisson algebra* P is a vector space endowed with a commutative multiplication and a Lie bracket $\{,\}$ that induces a derivation on the algebra in the form

$$\{a, bc\} = \{a, b\}c + b\{a, c\} \qquad \forall a, b, c \in P. \tag{2.9}$$

If P and Q are Poisson algebras, we can define the following Poisson structure on $P \otimes Q$:

$$\{a \otimes b, c \otimes d\}_{P\otimes Q} := \{a, c\}_P \otimes bd + ac \otimes \{b, d\}_Q. \tag{2.10}$$

We shall say that (A, Δ) is a *Poisson coalgebra* if A is a Poisson algebra and the coproduct Δ is a Poisson algebra homomorphism between A and $A \otimes A$:

$$\{\Delta_A(a), \Delta_A(b)\}_{A\otimes A} = \Delta(\{a, b\}_A) \qquad \forall a, b \in A. \tag{2.11}$$

Obviously, given any Lie algebra g a Poisson coalgebra can be obtained by defining a Poisson bracket by means of the bivector

$$\Lambda = c_{ij}^k x_k \partial_{x_i} \wedge \partial_{x_j} \tag{2.12}$$

where the x are local coordinates on a certain manifold linked to the generators of g and c_{ij}^k is the structure tensor for g. We can immediately check that the coproduct (2.4) is a Poisson map if the Poisson bracket on the tensor product space is defined by (2.10). Quantum deformations can also be realized as Poisson coalgebras in this way: a natural Poisson coalgebra linked to $U_z(so(2,1))$ is given by the bivector

$$\Lambda = \tilde{\sigma}_3 \partial_{\tilde{\sigma}_2} \wedge \partial_{\tilde{\sigma}_1} - \tilde{\sigma}_1 \partial_{\tilde{\sigma}_2} \wedge \partial_{\tilde{\sigma}_3} + \frac{\sinh(z\tilde{\sigma}_2)}{z} \partial_{\tilde{\sigma}_3} \wedge \partial_{\tilde{\sigma}_1} \tag{2.13}$$

and the coproduct (2.6) where the quantum algebra generators are replaced by their corresponding local coordinates $\tilde{\sigma}_i$ on $\mathbb{R}^3$. Obviously, the Poisson structure (2.13) wil be non-degenerate on the symplectic leaf defined by

$$\left(2\frac{\sinh(\frac{z}{2}\tilde{\sigma}_2)}{z}\right)^2 - \tilde{\sigma}_1^2 - \tilde{\sigma}_3^2 = c_z. \tag{2.14}$$

On the other hand, the connection between a Lie algebra and a one-particle system can be made explicit by considering that g is realized by means of smooth functions on the one-particle phase space $\mathbb{R}^2$ with local coordinates (p, q)

$$D(X_i) = X_i(p, q). \tag{2.15}$$

This means that, under the 'canonical' Poisson bracket

$$\{f, h\} = \frac{\partial f}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p} \qquad f, h \in C^\infty(p, q) \tag{2.16}$$

the 'generators' (2.15) close the initial Lie algebra:

$$\{X_i(p, q), X_j(p, q)\} = c_{ij}^k X_k(p, q). \tag{2.17}$$

Two different one-particle realizations (2.15) will be equivalent if there exists a canonical transformation that maps one into the other. A simple example is given by the following one-particle realization of the Poisson coalgebra linked to $so(2, 1)$:

$$D(J_2) = p \qquad D(J_1) = p\cos q \qquad D(J_3) = p\sin q. \tag{2.18}$$

This realization (that leads to a vanishing Casimir function) can be easily deformed:

$$D_z(\tilde{J}_2) = p \qquad D_z(\tilde{J}_1) = 2\frac{\sinh(\frac{z}{2}p)}{z}\cos q \qquad D_z(\tilde{J}_3) = 2\frac{\sinh(\frac{z}{2}p)}{z}\sin q. \tag{2.19}$$

These phase-space functions close a quantum $so(2, 1)$ algebra (2.7) under the canonical Poisson bracket (2.16).

Now, the essential feature of a Poisson coalgebra becomes evident: if we represent $A \otimes A$ by using two copies of (2.15), the functions $\Delta(X_i)(q_1, q_2, p_1, p_2)$ (we use the notation $p \otimes 1 \equiv p_1$, $1 \otimes p \equiv p_2$, and so on) define the same Lie algebra g

$$\{\Delta(X_i), \Delta(X_j)\}_{A\otimes A} = \Delta(\{X_i, X_j\}_A) = c_{ij}^k \Delta(X_k) \qquad \forall X_i, X_j \tag{2.20}$$

with respect to a bracket (2.20) given by

$$\{f, h\} = \sum_{i=1}^{2}\left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i}\right). \tag{2.21}$$

In particular, (2.21) leads to the Poisson bracket (2.10) provided we have chosen $f = a(q_1, p_1)b(q_2, p_2)$ and $h = c(q_1, p_1)d(q_2, p_2)$.

In the case of $so(2, 1)$, this coalgebra property means that the following two-particle functions defined through the coproduct (2.4) and the realization (2.18)

$$\begin{aligned} f_1(\boldsymbol{q}, \boldsymbol{p}) &= (D \otimes D)(\Delta(J_1)) = p_1\cos q_1 + p_2\cos q_2 \\ f_2(\boldsymbol{q}, \boldsymbol{p}) &= (D \otimes D)(\Delta(J_2)) = p_1 + p_2 \\ f_3(\boldsymbol{q}, \boldsymbol{p}) &= (D \otimes D)(\Delta(J_3)) = p_1\sin q_1 + p_2\sin q_2 \end{aligned} \tag{2.22}$$

close the $so(2,1)$ algebra. The deformed construction is also immediate: from (2.6) and (2.19) we obtain the functions

$$
\begin{aligned}
f_1^z(\boldsymbol{q},\boldsymbol{p}) &= (D_z\otimes D_z)(\Delta(\tilde{J}_1)) = 2\frac{\sinh(\frac{z}{2}p_1)}{z}\cos q_1 \mathrm{e}^{\frac{z}{2}p_2} + \mathrm{e}^{-\frac{z}{2}p_1} 2\frac{\sinh(\frac{z}{2}p_2)}{z}\cos q_2 \\
f_2^z(\boldsymbol{q},\boldsymbol{p}) &= (D_z\otimes D_z)(\Delta(\tilde{J}_2)) = p_1+p_2 \qquad (2.23)\\
f_3^z(\boldsymbol{q},\boldsymbol{p}) &= (D_z\otimes D_z)(\Delta(\tilde{J}_3)) = 2\frac{\sinh(\frac{z}{2}p_1)}{z}\sin q_1 \mathrm{e}^{\frac{z}{2}p_2} + \mathrm{e}^{-\frac{z}{2}p_1} 2\frac{\sinh(\frac{z}{2}p_2)}{z}\sin q_2
\end{aligned}
$$

that close a $U_z(so(2,1))$ algebra under the canonical Poisson bracket (2.21).

3. Casimirs and $N=2$ integrable systems

Let us fix our attention on the examples of the previous section. If we recall the deformed Casimir element (2.8) and its non-deformed counterpart

$$C(J_1,J_2,J_3) = J_2^2 - J_1^2 - J_3^2 \qquad (3.1)$$

we know that both elements vanish, respectively, under the realizations (2.19) and (2.18) (different canonical realizations will be labelled by the real value obtained when the Casimir is represented). However, if in the non-deformed $so(2,1)$ case we compute the coproduct of the Casimir (3.1), we obtain

$$
\begin{aligned}
\Delta(C) &= C(\Delta(J_1),\Delta(J_3),\Delta(J_2)) \\
&= (1\otimes J_2 + J_2\otimes 1)^2 - (1\otimes J_1 + J_1\otimes 1)^2 - (1\otimes J_3 + J_3\otimes 1)^2 \\
&= 1\otimes C + C\otimes 1 + 2(J_2\otimes J_2 - J_1\otimes J_1 - J_3\otimes J_3). \qquad (3.2)
\end{aligned}
$$

When this abstract object is realized by using the D representation we obtain

$$
\begin{aligned}
C^{(2)}(q_1,q_2,p_1,p_2) &\equiv (D\otimes D)(\Delta(C)) \\
&= 0 + 0 + 2[p_1p_2 - (p_1\cos q_1)(p_2\cos q_2) - (p_1\sin q_1)(p_2\sin q_2)] \\
&= 2p_1p_2(1-\cos(q_1-q_2)). \qquad (3.3)
\end{aligned}
$$

Therefore, although the Casimir vanishes on each space, the coproduct of C has a ‘crossed’ contribution that is not trivial in the two-particle realization.

This non-trivial nature of $\Delta(C)$ is the cornerstone for the systematic generation of a wide class of two-dimensional integrable systems; in any (Poisson) coalgebra endowed with a Casimir element, since the coproduct is an algebra homomorphism and C is a central element within $U(g)$, we can conclude that

$$\{\Delta(C),\Delta(X_i)\}_{A\otimes A} = \Delta(\{C,X_i\}_A) = 0 \qquad \forall X_i. \qquad (3.4)$$

Consequently, if the Hamiltonian $\mathcal{H}(X_1,\dots,X_m)$ is an arbitrary (smooth) function of the algebra generators we shall have that

$$\{\Delta(C),\Delta(\mathcal{H}(X_1,\dots,X_m))\}_{A\otimes A} = \Delta(\{C,\mathcal{H}(X_1,\dots,X_m)\}_A) = 0. \qquad (3.5)$$

Therefore, a canonical realization of the coproduct of any (smooth) function $\mathcal{H}$ of the algebra generators of a coalgebra with Casimir element C defines a two-particle completely integrable Hamiltonian. In our case, any Hamiltonian

$$
\begin{aligned}
H^{(2)}(q_1,q_2,p_1,p_2) &:= (D\otimes D)(\Delta(\mathcal{H}(J_1,J_2,J_3))) \\
&= (D\otimes D)(\mathcal{H}(\Delta(J_1),\Delta(J_2),\Delta(J_3))) = \mathcal{H}(f_1,f_2,f_3) \qquad (3.6)
\end{aligned}
$$

will always be in involution with the function $C(f_1, f_2, f_3)$ (3.3). For instance, the function

$$\mathcal{H} = J_2^2 + \kappa_2 J_1^2 + \kappa_1 J_3^2 \tag{3.7}$$

where κ_1 and κ_2 are real parameters (that have a precise geometrical meaning in the context of pseudo-orthogonal algebras [28]), together with the D realization and the formula (3.6) gives rise to the two-particle Hamiltonian

$$\begin{aligned} H^{(2)}(q_1, q_2, p_1, p_2) = (p_1 + p_2)^2 + 2p_1 p_2(\kappa_2 \cos q_1 \cos q_2 + \kappa_1 \sin q_1 \sin q_2) \\ + p_1^2(\kappa_2 \cos^2 q_1 + \kappa_1 \sin^2 q_1) + p_2^2(\kappa_2 \cos^2 q_2 + \kappa_1 \sin^2 q_2) \end{aligned} \tag{3.8}$$

that defines a two-parameter family of integrable systems for which (3.3) is a common constant of motion. If we specialize $\kappa_1 = \kappa_2 = 1$, we obtain

$$H^{(2)}(q_1, q_2, p_1, p_2) = 2(p_1^2 + p_2^2 + p_1 p_2(1 + \cos(q_1 - q_2)). \tag{3.9}$$

At this point, some remarks are in order.

(a) The choice of the Hamiltonian is constrained by the requirement of functional independence between the two constants of the motion. In particular, if we choose $\kappa_1 = \kappa_2 = -1$ we shall recover the coproduct of the Casimir $2p_1 p_2(1 - \cos(q_1 - q_2))$, but now playing the role of the Hamiltonian. However, integrability is now ensured by taking the coproduct of any generator as the second constant of the motion (if f_2, we deduce the conservation of the total momenta $p_1 + p_2$). Note that, in general, the coproduct of a given generator is not in involution with (3.8).

(b) Many different Hamiltonians may have the same 'hidden' coalgebra symmetry, since different phase-space representations and choices of the Hamiltonian function are possible.

3.1. $N = 2$ integrable deformations

Now it is essential to stress that the integrable nature of this construction is preserved for any possible coalgebra with Casimir element that we could consider. Of course, deformations of Lie algebras with coalgebra structure fall into this class and, therefore, can be used to construct integrable systems.

Moreover, if a Hamiltonian $H^{(2)}$ can be constructed by using the previous procedure, any coalgebra deformation of its symmetry algebra will generate an integrable deformation $H_z^{(2)}$ of $H^{(2)}$ (provided that a deformed Casimir element C_z and a deformed canonical realization D_z are available).

In particular, the standard quantum deformation of $so(2,1)$ (2.6)–(2.8) can be used to define integrable two-particle Hamiltonians through the deformed coproduct of an arbitrary function $\mathcal{H}$ of the generators:

$$\begin{aligned} H_z^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)(\Delta(\mathcal{H}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3))) \\ = (D_z \otimes D_z)(\mathcal{H}(\Delta(\tilde{J}_1), \Delta(\tilde{J}_2), \Delta(\tilde{J}_3))) = \mathcal{H}(f_1^z, f_2^z, f_3^z). \end{aligned} \tag{3.10}$$

This deformed Hamiltonian will always be in involution with the (deformed) phase-space representation of the coproduct of the deformed Casimir (2.8), that reads,

$$C_z^{(2)}(q_1, q_2, p_1, p_2) \equiv (D_z \otimes D_z)(\Delta(C_z)) = \pi_1 \pi_2 (1 - \cos(q_1 - q_2)) \tag{3.11}$$

where

$$\pi_1 = 2\frac{\sinh(\frac{z}{2}p_1)}{z}\mathrm{e}^{\frac{z}{2}p_2} \qquad \pi_2 = 2\frac{\sinh(\frac{z}{2}p_2)}{z}\mathrm{e}^{-\frac{z}{2}p_1}. \tag{3.12}$$

An example of such a deformed Hamiltonian is provided by (3.7) where the generators are now replaced by their deformed counterparts

$$\mathcal{H} = \tilde{J}_2^2 + \kappa_2 \tilde{J}_1^2 + \kappa_1 \tilde{J}_3^2. \tag{3.13}$$

From (3.10), and by making use of the deformed phase-space realization (2.23), we obtain the integrable family of Hamiltonians

$$\begin{aligned} H_z^{(2)}(q_1, q_2, p_1, p_2) &= (f_2^z)^2 + \kappa_2 (f_1^z)^2 + \kappa_1 (f_3^z)^2 \\ &= (p_1 + p_2)^2 + 2\pi_1\pi_2(\kappa_2 \cos q_1 \cos q_2 + \kappa_1 \sin q_1 \sin q_2) \\ &\quad + \pi_1^2(\kappa_2 \cos^2 q_1 + \kappa_1 \sin^2 q_1) + \pi_2^2(\kappa_2 \cos^2 q_2 + \kappa_1 \sin^2 q_2). \end{aligned} \tag{3.14}$$

Now, the deformation of the particular case $\kappa_1 = \kappa_2 = 1$ reads

$$H^{(2)}(q_1, q_2, p_1, p_2) = (p_1 + p_2)^2 + \pi_1^2 + \pi_2^2 + 2\pi_1\pi_2 \cos(q_1 - q_2). \tag{3.15}$$

Note that, after deformation, the case ($\kappa_1 = \kappa_2 = -1$) is no longer the realization of the deformed Casimir (3.11). Of course, in order to obtain the Casimir as a Hamiltonian we should consider $\mathcal{H} \equiv C_z$; then, any f_z^i can be taken as the remaining integral of the motion in involution. It also becomes apparent that the limit $z \to 0$ of (3.14) is just (3.8).

4. Coassociativity and higher-order coproducts

The coassociativity constraint (2.1) on Δ means that, in principle, we could extend the previous procedure in order to obtain a more complex system with three elementary constituents. If we denote $\Delta \equiv \Delta^{(2)}$ (in order to make more explicit the fact that Δ defines a two-particle system) the mapping $\Delta^{(3)} : A \to A \otimes A \otimes A$ has to be defined through (2.1) by using one of the following expressions:

$$\begin{aligned} \Delta^{(3)} &:= (\mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(2)} \\ \Delta^{(3)} &:= (\Delta^{(2)} \otimes \mathrm{id}) \circ \Delta^{(2)}. \end{aligned} \tag{4.1}$$

From (2.1), the result of this procedure is unique and does not depend on the space within $A \otimes A$ we had chosen to duplicate.

On the other hand, it is well known that, once the coassociativity has ensured the correctness of the three-constituents system, the construction can be generalized to an arbitrary number of tensor products of A. For instance, we would have that

$$\Delta^{(4)} := (\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(3)} \tag{4.2}$$

will give rise to a fourth-order coproduct starting from the third one. In general, this procedure is described in the literature either by the recurrence relation

$$\Delta^{(N)} := (\mathrm{id} \otimes \mathrm{id} \otimes \ldots^{N-2)} \otimes \mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(N-1)} \tag{4.3}$$

or by the following similar one

$$\Delta^{(N)} := (\Delta^{(2)} \otimes \mathrm{id} \otimes \ldots^{N-2)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(N-1)}. \tag{4.4}$$

These definitions mean that given the $(N-1)$th coproduct, the Nth one is obtained by applying $\Delta^{(2)}$ onto the space located at the very right (resp. left) site. As a consequence, both (4.3) and (4.4) always emphasize the role of such 'boundary' vector spaces within the tensor product. This should not be necessary, since the essential meaning of coassociativity is that all elementary spaces are equivalent in order to build up a larger representation space by using the coproduct.

The algebraic transcription of this simple observation is the keystone for all further developements included in this paper, and both the recurrence character of (4.3) and its just mentioned 'asymmetry' can be avoided by means of the following definition

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \circ \Delta^{(2)} \qquad \forall m = 1, \ldots, N-1 \tag{4.5}$$

where $\Delta^{(1)}$ denotes the identity map id. The proof of the equivalence between (4.5) and the usual one (4.3) is given in appendix A. The meaning of this new expression (4.5) can be made more clear with the use of Sweedler's notation [29] that expresses the two-coproduct $\Delta^{(2)}$ of an arbitrary element of the algebra as the linear combination

$$\Delta^{(2)}(X) = \sum_{\alpha} X_{1\alpha} \otimes X_{2\alpha} \tag{4.6}$$

where $X_{1\alpha}$ and $X_{2\alpha}$ are functions depending on the generators of the algebra. By introducing this language in (4.5) we find that the the Nth coproduct of a generator reads

$$\Delta^{(N)}(X) := \sum_{\alpha} \Delta^{(m)}(X_{1\alpha}) \otimes \Delta^{(N-m)}(X_{2\alpha}) \qquad \forall m = 1, \ldots, N-1 \tag{4.7}$$

which means that the final result can be obtained in $N-1$ different ways, all of them equivalent, and given by the simultaneous application of two lower-degree coproducts on each of the two tensor components produced by $\Delta^{(2)}$.

Now it is not difficult to prove, by induction, that $\Delta^{(N)}$ is an algebra homomorphism between A and $A^{\otimes N}$

$$[\Delta^{(N)}(X), \Delta^{(N)}(Y)\}_{A^{\otimes N}} = \Delta^{(N)}([X, Y\}_A) \qquad \forall X, Y \in A. \tag{4.8}$$

A proof for this assertion can be found in appendix B. It is important to stress that the symbol $[x, y\}$ denotes a general bracket, that can be either the Poisson bracket for classical systems or the usual commutator for quantum mechanical ones. The underlying algebraic structure is the same for both kinds of systems and the differences existing between them arise from the different representation spaces we are working in.

In particular, in the case of $A \equiv U(g)$ and $\Delta^{(2)}$ given by (2.4), the following Nth coproduct for the generators of g is obtained:

$$\begin{aligned}\Delta^{(N)}(X_i) = X_i \otimes 1 \otimes 1 \otimes \ldots^{N-1)} \otimes 1 + 1 \otimes X_i \otimes 1 \otimes \ldots^{N-2)} \otimes 1 + \cdots \\ +1 \otimes 1 \otimes \ldots^{N-1)} \otimes 1 \otimes X_i.\end{aligned} \tag{4.9}$$

which is just the definition of the usual 'total observable', and for which the homomorphism condition is obviously fulfilled.

A more interesting example is provided by the deformation of (4.9) induced from (2.6). An iterative use of (4.3) leads to the following expressions

$$\begin{aligned}\Delta^{(N)}(\tilde{J}_2) &= \tilde{J}_2 \otimes 1 \otimes 1 \otimes \ldots^{N-1)} \otimes 1 + 1 \otimes \tilde{J}_2 \otimes 1 \otimes \ldots^{N-2)} \otimes 1 + \cdots \\ &\quad +1 \otimes 1 \otimes \ldots^{N-1)} \otimes 1 \otimes \tilde{J}_2 \\ \Delta^{(N)}(\tilde{J}_i) &= \tilde{J}_i \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \otimes \ldots^{N-1)} \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \\ &\quad +\mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \tilde{J}_i \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \otimes \ldots^{N-2)} \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} + \cdots \\ &\quad +\mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \ldots^{N-1)} \otimes \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \tilde{J}_i \qquad i = 1, 3.\end{aligned} \tag{4.10}$$

Now, by taking into account that, if X is a primitive generator and h is an arbitrary complex parameter, the relation $\Delta(\mathrm{e}^{hX}) = \mathrm{e}^{hX} \otimes \mathrm{e}^{hX}$ holds, we can choose any integer m running from 1 to $N-1$ and check that (4.10) can be written in a much more compact form

$$\begin{aligned}\Delta^{(N)}(\tilde{J}_i) &= \Delta^{(m)}(\tilde{J}_i) \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} \otimes \ldots^{N-m)} \otimes \mathrm{e}^{\frac{z}{2}\tilde{J}_2} + \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \ldots^{m)} \otimes \mathrm{e}^{-\frac{z}{2}\tilde{J}_2} \otimes \Delta^{(N-m)}(\tilde{J}_i) \\ &= \Delta^{(m)}(\tilde{J}_i) \otimes \mathrm{e}^{\frac{z}{2}\Delta^{(N-m)}(\tilde{J}_2)} + \mathrm{e}^{-\frac{z}{2}\Delta^{(m)}(\tilde{J}_2)} \otimes \Delta^{(N-m)}(\tilde{J}_i)\end{aligned} \tag{4.11}$$

that exactly corresponds to the result that we would have obtained by directly applying (4.5). This expression was already used in [18] to demonstrate the integrability of a precise system constructed from the standard deformation of $so(2,1)$.

5. The construction of N-particle Hamiltonians

The procedure to obtain $N=2$ integrable systems presented in section 3 can be generalized to any number of degrees of freedom by making use of the Nth coproduct. The statements here presented are valid for both classical (Poisson) and quantum mechanical (commutator) realizations of the underlying coalgebra (A,Δ). In order to emphasize this fact, the symbol $[x,y\}$ will be used hereafter; appendix B contains the computations that support this notation. On the other hand, the usual embedding of $A\otimes A\otimes\ldots^{m)}\otimes A$ within $A\otimes A\otimes\ldots^{N)}\otimes A$ as

$$A\otimes A\otimes\ldots^{m)}\otimes A\otimes 1\otimes\ldots^{N-m)}\otimes 1 \tag{5.1}$$

will be applied.

5.1. General results

The following proposition holds.

Proposition 1. Let (A,Δ) be a coalgebra with generators $X_i, i=1,\ldots,l$ and Casimir element $C(X_1,\ldots,X_l)$, and let us consider the Nth coproduct $\Delta^{(N)}(X_i)$ of the generators and the mth coproduct $\Delta^{(m)}(C)$ of the Casimir. Then,

$$[\Delta^{(m)}(C),\Delta^{(N)}(X_i)\}_{A\otimes A\otimes\ldots^{N)}\otimes A}=0 \qquad i=1,\ldots,l \qquad 1\leqslant m\leqslant N. \tag{5.2}$$

Proof. The case $m=N$ is easily proven by applying the homomorphism property for the Nth coproduct. On the other hand, by following Sweedler's notation, the second coproduct of the Casimir can be written as the sum

$$\Delta^{(2)}(C)=\sum_\alpha C_{1\alpha}\otimes C_{2\alpha}. \tag{5.3}$$

If we now compute (5.2) we obtain

$$[\Delta^{(m)}(C),\Delta^{(N)}(X_i)\}_{A\otimes\ldots^{N)}\otimes A}=[\Delta^{(m)}(C)\otimes 1\otimes\ldots^{N-m)}\otimes 1,\Delta^{(N)}(X_i)\}_{A\otimes\ldots^{N)}\otimes A} \tag{5.4}$$

$$=[\Delta^{(m)}(C)\otimes 1\otimes\ldots^{N-m)}\otimes 1,(\Delta^{(m)}\otimes\Delta^{(N-m)})\circ\Delta^{(2)}(X_i)\}_{A\otimes\ldots^{N)}\otimes A} \tag{5.5}$$

$$=\sum_\alpha[\Delta^{(m)}(C)\otimes 1\otimes\ldots^{N-m)}\otimes 1,\Delta^{(m)}(C_{1\alpha})\otimes\Delta^{(N-m)}(C_{2\alpha})\}_{A\otimes\ldots^{N)}\otimes A} \tag{5.6}$$

$$=\sum_\alpha[\Delta^{(m)}(C),\Delta^{(m)}(C_{1\alpha})\}_{A\otimes\ldots^{m)}\otimes A}\otimes\Delta^{(N-m)}(C_{2\alpha}) \tag{5.7}$$

$$=\sum_\alpha\Delta^{(m)}([C,C_{1\alpha}\}_A)\otimes\Delta^{(N-m)}(C_{2\alpha})=0 \tag{5.8}$$

where (5.4) reflects the usual embedding (5.1). The next step (5.5) includes the definition (4.5), that is applied in (5.6) with the help of (5.3). At this point the identity functions in the first term allow us to split the (Poisson/commutator) bracket as (5.7), and the fact that we have considered the mth coproducts for the Casimirs leads to the final result by taking into account that any order coproduct is a (Poisson/commutator) map and that $[C,C_{1\alpha}\}_A=0$ for any $C_{1\alpha}$ function. □

This result provides a straightforward generalization of the $N=2$ construction of integrable systems sketched in section 3.

Theorem 2. Let (A, Δ) be a coalgebra with generators X_i, $i = 1, \ldots, l$ and Casimir element $C(X_1, \ldots, X_l)$ and let $\mathcal{H}$ be an arbitrary (smooth/formal power series) function of the generators of A. Then, the N-particle Hamiltonian

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l)) = \mathcal{H}(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)) \tag{5.9}$$

fulfils

$$[C^{(m)}, H^{(N)}\}_{A\otimes A\otimes \ldots^{N)}\otimes A} = 0 \qquad 1 \leqslant m \leqslant N, \tag{5.10}$$

where the N Casimir elements $C^{(m)}$ $(m = 1, \ldots, N)$ are defined through

$$C^{(m)} := \Delta^{(m)}(C(X_1, \ldots, X_l)) = C(\Delta^{(m)}(X_1), \ldots, \Delta^{(m)}(X_l)). \tag{5.11}$$

Proof. The fact that $H^{(N)}$ and $C^{(N)}$ are in involution is again a straightforward consequence of the homomorphism property of $\Delta^{(N)}$. The rest of the proof follows directly from proposition 1, that tell us that the Nth coproduct of any generator commutes with all the lower-dimensional coproducts of the Casimir. Since our $\mathcal{H}$ is an arbitrary function of such generators it will (Poisson)-commute with all the $\Delta^{(m)}(C)$ elements. □

Corollary 3. In particular, all the $C^{(i)}$ elements generated by the Casimirs are in involution

$$[\Delta^{(k)}(C), \Delta^{(j)}(C)\} = 0 \qquad \forall k, j. \tag{5.12}$$

To prove this assertion, it suffices to take $N = \max\{k, j\}$ and apply the theorem in the case $\mathcal{H} \equiv C$. This ensures the involutivity among all the constants of motion. Note that, in principle, we have a set of $N+1$ constants of motion $\{C^{(1)}, C^{(2)}, \ldots, C^{(N)}, H^{(N)}\}$, but $C^{(1)}$ can be a real number (see the examples in section 3) and, in that case, we are left with N non-trivial integrals. On the other hand, functional independence among them is guaranteed by the fact that each $C^{(i)}$ element lives on $A \otimes A \otimes \ldots^{i)} \otimes A$ and that only $C^{(N)}$ and $H^{(N)}$ will share the same tensor space. In case $H^{(N)}$ is functionally dependent on $C^{(N)}$, we can always take the Nth coproduct of any generator as the remaining independent constant of motion.

It is also immediate to check that, if our coalgebra has more than one functionally independent Casimir elements C_i, the previous results hold simultaneously for all of them.

5.2. Classical mechanical systems

The systematic construction of classical systems is provided by the previous results when applied onto a Poisson coalgebra. Complete integrability is obtained when a canonical realization D of the Poisson coalgebra is added to the general algebraic construction. As a consequence, under such D, the Poisson bracket to be used is

$$\{f, h\}_{A\otimes A\otimes \ldots^{N)}\otimes A} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i} \right) \tag{5.13}$$

the N-particle classical Hamiltonian is written

$$\begin{aligned} H^{(N)}(q_1, \ldots, q_N, p_1, \ldots, p_N) &:= (D \otimes \ldots^{N)} \otimes D)(\Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l))) \\ &= (D \otimes \ldots^{N)} \otimes D)(\mathcal{H}(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l))) \\ &= \mathcal{H}((D \otimes \ldots^{N)} \otimes D)(\Delta^{(N)}(X_1)), \ldots, (D \otimes \ldots^{N)} \otimes D)(\Delta^{(N)}(X_l))) \end{aligned} \tag{5.14}$$

and the $N-1$ Casimir functions $C^{(m)}$ $(m=1,\ldots,N)$ read

$$
\begin{aligned}
C^{(m)}(q_1,\ldots,q_N,p_1,\ldots,p_N) &:= (D\otimes\ldots^{m)}\otimes D)(\Delta^{(m)}(C(X_1,\ldots,X_l)))\\
&= (D\otimes\ldots^{m)}\otimes D)(C(\Delta^{(m)}(X_1),\ldots,\Delta^{(m)}(X_l)))\\
&= C((D\otimes\ldots^{m)}\otimes D)(\Delta^{(m)}(X_1)),\ldots(D\otimes\ldots^{m)}\otimes D)(\Delta^{(m)}(X_l))).
\end{aligned}
\tag{5.15}
$$

Since each space is linked to only one degree of freedom, complete integrability of the Nth Hamiltonian follows from (5.15). If we are dealing with r independent Casimir functions C_i, the formalism can lead to the preservation of complete integrability for any realization D depending on t pairs $(t\leqslant r)$ of canonical coordinates. Moreover, nothing prevents us from the use of 'non-canonical' realizations, as we shall see in the following. On the other hand, in the case that other canonical realizations $D', D'', \ldots$ exist, their simultaneous use in order to realize the tensor products of A as, for instance, $D\otimes D'\otimes D''\ldots$ will provide 'mixed' realizations of the same underlying abstract coalgebra.

It is also important to stress that no assumption concerning the explicit form of the coproduct is needed to prove these statements. Therefore, deformed Poisson coalgebras can be implemented with no difficulty within this algorithm in order to provide (deformed) integrable systems, as was done for $N=2$ in section 3.1.

5.3. Quantum mechanical systems

Proofs of the aforementioned results when the commutator bracket is considered, offer no particular comments, up to those already included in appendix B, and the essential algebraic features of the general method presented here are not modified by the non-commutativity of the algebra A with respect to the $(\cdot)$ product.

However, from a computational point of view it is important to stress that in general extra contributions coming from the unavoidable reordering processes will have to be considered. Likewise, the quantum mechanical analogues of canonical realizations D will be obtained either by using the generators $\hat{p}$ and $\hat{q}$ of the Heisenberg–Weyl algebra or by means of the so-called boson realizations in terms of the operators a and a_+ fulfilling $[a,a_+]=1$ (see [30] and the references there included for recent applications of bosonization procedures in the representation theory of quantum algebras). Among the Hamiltonians that are explicitly constructed in what follows, those expresed in terms of canonical coordinates should be quantized in that way, and the remaining ones would lead to quantum angular momentum (and, in particular, spin) chains.

6. Some coalgebra-invariant classical integrable systems

We now present some examples of completely integrable systems obtained with the aid of the previous results. Some of them are (to our knowledge) new ones, and others (although already known) are shown to underly a 'hidden' coalgebra symmetry. Integrable deformations appear under quantum coalgebra symmetry in a direct way.

6.1. A $so(2,1)$ family including Calogero systems

If we recall the (undeformed) Nth coproduct (4.9) for the $so(2,1)$ Poisson coalgebra and consider N copies $D\otimes D\ldots^{N)}\otimes D$ of the canonical phase-space realization (2.18) we obtain the following N-particle functions

$$
\begin{aligned}
f_1(\boldsymbol{q},\boldsymbol{p}) &= (D\otimes D\ldots^{N)}\otimes D)(\Delta^{(N)}(J_1)) = \sum_{i=1}^{N} p_i\cos q_i \\
f_2(\boldsymbol{q},\boldsymbol{p}) &= (D\otimes D\ldots^{N)}\otimes D)(\Delta^{(N)}(J_2)) = \sum_{i=1}^{N} p_i \\
f_3(\boldsymbol{q},\boldsymbol{p}) &= (D\otimes D\ldots^{N)}\otimes D)(\Delta^{(N)}(J_3)) = \sum_{i=1}^{N} p_i\sin q_i
\end{aligned}
\tag{6.1}
$$

that close an $so(2,1)$ coalgebra. Now, if we take as Hamiltonian function $\mathcal{H}$ the quadratic two-parameter function (3.7), theorem 2 gives rise to the following integrable Hamiltonian

$$
H^{(N)}(\boldsymbol{q},\boldsymbol{p}) = \left(\sum_{i=1}^{N} p_i\right)^2 + \tfrac{1}{2}\sum_{i,j=1}^{N} p_i p_j((\kappa_1+\kappa_2)\cos(q_i-q_j) - (\kappa_1-\kappa_2)\cos(q_i+q_j)).
\tag{6.2}
$$

All the constants of motion in involution are given by the phase-space realizations of all the mth coproducts of the Casimir function $C = J_2^2 - J_1^2 - J_3^2$ (see (5.15)), that read

$$
C^{(m)}(\boldsymbol{q},\boldsymbol{p}) = \sum_{i<j}^{m} 2p_i p_j(1-\cos(q_i-q_j)) \qquad m = 2,\ldots,N.
\tag{6.3}
$$

Note that, in the chosen realization, $C^{(1)}(q_1, p_1) = 0$.

The case $\kappa_1 = \kappa_2$ means that (6.2) depends on the differences $(q_i - q_j)$. In particular, if we specialize the parameters in the form $\kappa_1 = \kappa_2 = -1$, the chosen Hamiltonian coincides with the Casimir. In that case, (6.2) is just $C^{(N)}$ and (6.3) gives us $N-1$ constants of motion in involution (but, for instance, any f_i function (6.1) can be chosen to obtain a complete family of integrals). The system $H^{(N)} = \sum_{i<j}^{N} 2p_i p_j(1-\cos(q_i-q_j))$ was first introduced by Calogero [20] as an integrable Hamiltonian of the general type $H = \sum_{i<j}^{N} p_i p_j f(q_i - q_j)$ (the Hamiltonian structures underlying an integrable nonlinear shallow-water equation with peaked solitons—the so-called 'peakons' [31]—belongs to that class of systems).

The 'hidden coalgebra symmetry' of this particular system was explicitly introduced in [18] and it was also implicitely stated in [19]. However, a crucial point is that any function $\mathcal{H}$ of the generators (and not only the Casimirs) can now be taken as the (integrable) Hamiltonian, thus generalizing the original Calogero model in a highly arbitrary way. For instance, if we specialize the parameters as $\kappa_1 = \kappa_2 = 1$, we arrive at the N-particle generalization of (3.9):

$$
H^{(N)}(\boldsymbol{q},\boldsymbol{p}) = \sum_{i=1}^{N} 2p_i^2 + \sum_{i<j}^{N} 2p_i p_j(1+\cos(q_i-q_j))
\tag{6.4}
$$

which is of course in involution with all the $C^{(m)}$ functions (6.3).

6.2. The algebra h_4 and an integrable oscillator chain

Any other Lie algebra can give rise to an integrable system by following the same procedure. For instance, we mention here the oscillator Lie algebra h_4 is generated by $\{N, A_+, A_-, M\}$ with Lie–Poisson brackets

$$
\{N, A_+\} = A_+ \qquad \{N, A_-\} = -A_- \qquad \{A_-, A_+\} = M \qquad \{M, \cdot\} = 0.
\tag{6.5}
$$

Besides the central generator M there exists another Casimir invariant for h_4:

$$C = NM - A_+A_-. \tag{6.6}$$

A canonical D realization for this algebra with vanishing Casimir C is given by

$$D(N) = p \qquad D(A_+) = \sqrt{p}\mathrm{e}^{-q} \qquad D(A_-) = \sqrt{p}\mathrm{e}^{q} \qquad D(M) = 1. \tag{6.7}$$

Let us now consider the $\mathcal{H}$ function

$$\mathcal{H} = \lambda N + \mu A_+A_-. \tag{6.8}$$

It is immediate to check that, by using the primitive coproduct (4.9) for all the generators, theorem 2 provides the following integrable Hamiltonian

$$H^{(N)}(\boldsymbol{q}, \boldsymbol{p}) = (\lambda + \mu)\sum_{i=1}^{N} p_i + 2\mu \sum_{i<j}^{N} \sqrt{p_i p_j}\cosh(q_i - q_j) \tag{6.9}$$

which is just the one introduced in [21]. The integrals of the motion in involution are given by the coproducts of the Casimir (6.6) in the chosen realization, and read

$$C^{(m)}(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{m} p_i - \sum_{i<j}^{m} 2\sqrt{p_i p_j}\cosh(q_i - q_j). \tag{6.10}$$

The quantization of the Hamiltonian (6.9) has been performed in [32], where the equivalence between the quantum version of (6.9) and a system of coupled oscillators is shown (see also [33]).

6.3. An integrable deformation from $U_z so(2,1)$

The (standard) quantum deformation of $so(2,1)$ generates, through the Nth order generalization of the comultiplication map (4.10) and the deformed realization D_z, an integrable deformation of the family (6.2). Let us fix N and start by defining the quantities

$$\pi_k = 2\frac{\sinh(\frac{z}{2}p_k)}{z}\left(\prod_{i=1}^{k-1}\mathrm{e}^{-\frac{z}{2}p_i}\right)\left(\prod_{j=k+1}^{N}\mathrm{e}^{\frac{z}{2}p_j}\right). \tag{6.11}$$

The N-particle canonical (deformed) phase-space realization will be

$$\begin{aligned} f_1^z(\boldsymbol{q}, \boldsymbol{p}) &= (D_z \otimes D_z \ldots^{N)} \otimes D_z)(\Delta^{(N)}(\tilde{J}_1)) = \sum_{i=1}^{N} \pi_i \cos q_i \\ f_2^z(\boldsymbol{q}, \boldsymbol{p}) &= (D_z \otimes D_z \ldots^{N)} \otimes D_z)(\Delta^{(N)}(\tilde{J}_2)) = \sum_{i=1}^{N} p_i \\ f_3^z(\boldsymbol{q}, \boldsymbol{p}) &= (D_z \otimes D_z \ldots^{N)} \otimes D_z)(\Delta^{(N)}(\tilde{J}_3)) = \sum_{i=1}^{N} \pi_i \sin q_i. \end{aligned} \tag{6.12}$$

Now it is clear that, by taking as Hamiltonian function (3.13), theorem 2 provides the following integrable Hamiltonian

$$H^{(N)}(\boldsymbol{q}, \boldsymbol{p}) = \left(\sum_{i=1}^{N} p_i\right)^2 + \frac{1}{2}\sum_{i,j=1}^{N} \pi_i\pi_j((\kappa_1 + \kappa_2)\cos(q_i - q_j) - (\kappa_1 - \kappa_2)\cos(q_i + q_j)). \tag{6.13}$$

The integrals of the motion are just the D_z realization of the mth deformed coproducts of C_z (2.8). A closed expression for them can be readily obtained if we realize that the π_i functions fulfil the relation

$$\frac{2}{z}\sinh(\frac{z}{2}(p_1 + p_2 + \cdots + p_m)) = \pi_1 + \pi_2 + \cdots + \pi_m. \tag{6.14}$$

Now it is not difficult to check that the explicit formula for $C_z^{(m)}$ is

$$C_z^{(m)} = \sum_{i<j}^{m} 2\pi_i\pi_j(1 - \cos(q_i - q_j)) \qquad m = 1, \ldots, N \tag{6.15}$$

where and, as expected, in the limit $z \to 0$ we recover the 'classical' Gaudin–Calogero system (6.3).

Once again, the particular deformed system $\kappa_1 = \kappa_2 = -1$ does not coincide with the Nth Casimir function, although the former can be obtained from the latter by substracting the function $\Delta^{(N)}((2\frac{\sinh(\frac{z}{2}\tilde{J}_2)}{z})^2 - \tilde{J}_2^2)$.

6.4. A Ruijsenaars–Schneider-like model from a quantum deformation of (1 + 1) Poincaré algebra

The $(1+1)$ Poincaré algebra $\mathcal{P}(1,1)$ is generated by $\{K, H, P\}$ and can be realized in Poisson form by the following brackets

$$\{K, H\} = P \qquad \{K, P\} = H \qquad \{P, H\} = 0. \tag{6.16}$$

The known Casimir function for $\mathcal{P}(1,1)$ is,

$$C = H^2 - P^2 \tag{6.17}$$

and a $C = 1$ Poisson realization of this algebra in terms of a canonical coordinate q and its conjugate rapidity θ is the following:

$$D(K) = q \qquad D(H) = \cosh\theta \qquad D(P) = \sinh\theta. \tag{6.18}$$

If we consider the primitive coproduct (4.9) and take as Hamiltonian function just the H generator, the resultant coalgebra-induced integrable system reads,

$$H^{(N)}(\boldsymbol{q}, \boldsymbol{\theta}) = \sum_{i=1}^{N}\cosh\theta_i \qquad C^{(m)}(\boldsymbol{q}, \boldsymbol{\theta}) = m + \sum_{i<j}^{m} 2\cosh(\theta_i - \theta_j). \tag{6.19}$$

Note that the associated dynamics is quite trivial since (6.19) depends only on the canonical momenta.

However, a completely different system is derived when we consider the (non-coboundary) quantum deformation $U_z\mathcal{P}(1,1)$ given by the deformed coproduct

$$\begin{aligned} \Delta(K) &= 1 \otimes K + K \otimes 1 \\ \Delta(H) &= \mathrm{e}^{-\frac{z}{2}K} \otimes H + H \otimes \mathrm{e}^{\frac{z}{2}K} \\ \Delta(P) &= \mathrm{e}^{-\frac{z}{2}K} \otimes P + P \otimes \mathrm{e}^{\frac{z}{2}K} \end{aligned} \tag{6.20}$$

that, in spite of the non-triviality of the deformation, is still compatible with the undeformed brackets (6.16). This deformation was first introduced in [22], and it was later recognized as the dual of Woronowicz's quantum (pseudo-)Euclidean group [34].

Therefore, the compatibility with (6.16) implies that the phase-space realization (6.18) is also valid in the deformed case. If we consider again the time translation H as the

Hamiltonian function $\mathcal{H}$, the Nth generalization of the deformed coproduct and the phase-space realization (6.18) gives rise to the integrable system defined by

$$H_z^{(N)}(\boldsymbol{q},\boldsymbol{\theta}) = \sum_{i=1}^{N} \cosh\theta_i \exp\left(-\frac{z}{2}\left(\sum_{j=1}^{i-1} q_j\right) + \frac{z}{2}\left(\sum_{k=i+1}^{N} q_k\right)\right). \tag{6.21}$$

This system presents strong analogies with respect to the so-called Ruijsenaars–Schneider Hamiltonian [23], which is a relativistic analogue of Calogero–Moser systems.

The integrals of motion are obtained, as usual, from the Nth order deformed coproducts of the Casimir (6.17). A straightforward computation shows that they are

$$C_z^{(m)}(\boldsymbol{q},\boldsymbol{\theta}) = \sum_{i<j}^{m} 2\cosh(\theta_i-\theta_j) \exp\left(-\frac{z}{2}(q_i-q_j) - z\left(\sum_{l=1}^{i-1} q_l\right) + z\left(\sum_{k=j+1}^{N} q_k\right)\right). \tag{6.22}$$

Note that in this case additional integrals appear due to the fact that P commutes with H. In particular, the deformed Nth coproduct of P

$$P_z^{(N)}(\boldsymbol{q},\boldsymbol{\theta}) = \sum_{i=1}^{N} \sinh\theta_i \exp\left(-\frac{z}{2}\left(\sum_{j=1}^{i-1} q_j\right) + \frac{z}{2}\left(\sum_{k=i+1}^{N} q_k\right)\right) \tag{6.23}$$

will Poisson-commute with both $H_z^{(N)}$ and $C_z^{(N)}$.

7. Angular momentum realizations

The coalgebra symmetry that gives rise to N integrals of motion in involution is not restricted to the use of canonical realizations. We shall consider in this section its application to the construction of classical integrable 'angular momentum' chains through the $so(2,1)$ Poisson coalgebra given by a primitive coproduct and the Poisson bivector

$$\Lambda = \sigma_3\partial_{\sigma_2}\wedge\partial_{\sigma_1} - \sigma_1\partial_{\sigma_2}\wedge\partial_{\sigma_3} + \sigma_2\partial_{\sigma_3}\wedge\partial_{\sigma_1} \tag{7.1}$$

afterwards, its deformed counterpart (2.13) will be examined and the consequences of the deformation analysed.

These examples will also stress the possibilities of applying the actual formalism to the quantum mechanical context. From the following examples it will become clear that quantization will imply (up to sometimes important contributions coming from reordering) the substitution of the σ coordinates by the corresponding Pauli matrices. In this way, the $so(2,1)$ systems can be interpreted as Gaudin magnets, and the quantum deformation of the coalgebra will introduce a variable range interaction in the model. An exhaustive study of these aspects will be presented elsewhere.

7.1. The $so(2,1)$ model: Classical XYZ Gaudin magnet

Let us now consider the Poisson bracket (7.1) corresponding to the $so(2,1)$ Lie algebra, which is tantamount to considering—in our language—the realization S

$$S(J_2) = \sigma_2 \qquad S(J_1) = \sigma_1 \qquad S(J_3) = \sigma_3 \tag{7.2}$$

that will be completely defined provided the value $c = \sigma_2^2 - \sigma_1^2 - \sigma_3^2$ is given. Now, a straightforward replica of the generalized Calogero systems of the previous section is provided by N-copies of (7.2) (that we shall distinguish with the aid of a superindex σ_i^k

and that could have different values c_k of the Casimir) and the (undeformed) Nth coproduct (4.9). Therefore, we have the (quite trivial) realization for the coproducts

$$(S \otimes S \ldots^{N)} \otimes S)(\Delta^{(N)}(\sigma_i)) = \sum_{k=1}^{N} \sigma_i^k \qquad i = 1, 2, 3. \tag{7.3}$$

If we preserve (3.7) as a Hamiltonian function, theorem 2 provides the following integrable Hamiltonian:

$$\begin{aligned} H^{(N)}(\boldsymbol{\sigma}) &= \left(\sum_{l=1}^{N} \sigma_2^l \right)^2 + \kappa_2 \left(\sum_{l=1}^{N} \sigma_1^l \right)^2 + \kappa_1 \left(\sum_{l=1}^{N} \sigma_3^l \right)^2 \\ &= \sum_{l=1}^{N} \{(\sigma_2^l)^2 + \kappa_2 (\sigma_1^l)^2 + \kappa_1 (\sigma_3^l)^2\} + 2 \sum_{i<j}^{N} \{\sigma_2^i \sigma_2^j + \kappa_2 \sigma_1^i \sigma_1^j + \kappa_1 \sigma_3^i \sigma_3^j\}. \end{aligned} \tag{7.4}$$

That is, a classical long-range interacting XYZ angular momentum chain of the Gaudin type [24, 35, 36].

The constants of motion are derived from the mth coproducts of the Casimir function $C = J_2^2 - J_1^2 - J_3^2$ in the usual way and read,

$$C^{(m)}(\boldsymbol{\sigma}) = \sum_{l=1}^{m} c_l + 2 \sum_{i<j}^{m} \sigma_2^i \sigma_2^j - \sigma_1^i \sigma_1^j - \sigma_3^i \sigma_3^j. \tag{7.5}$$

Since the first term is constant, we are lead to the hyperbolic XXX-Gaudin system. Note that this system becomes the keystone for the integrability of any finite chain obtained through an arbitrary function of the $so(2,1)$ generators. On the other hand, this construction can be immediately quantized by transforming (7.2) into a representation in terms of angular momentum operators and by taking into account the corresponding discrete values for the Casimir operators.

7.2. $U_z(so(2,1))$ and XYZ model with variable range exchange

Let us now construct an integrable deformation of the XYZ classical Gaudin system through $U_z(so(2,1))$. The Poisson realization S_z that we are going to consider is

$$S_z(\tilde{J}_2) = \tilde{\sigma}_2 \qquad S_z(\tilde{J}_1) = \tilde{\sigma}_1 \qquad S_z(\tilde{J}_3) = \tilde{\sigma}_3 \tag{7.6}$$

with c_z given by (2.14). Note that the $\tilde{\sigma}_i$ coordinates are not the classical ones (they live on a deformed hyperboloid (2.14)), although we shall consider a particular representation in terms of the classical structure (7.1) later.

As usual, the comultiplication map (4.10) and the chosen realization S_z gives rise to the following functions expressing the Nth-order coproduct of the $\tilde{J}_i$ generators:

$$\begin{aligned} (S_z \otimes S_z \ldots^{N)} \otimes S_z)(\Delta^{(N)}(\tilde{J}_1)) &= \sum_{l=1}^{N} \left(\prod_{i=1}^{l-1} \mathrm{e}^{-\frac{z}{2}\tilde{\sigma}_2^i} \right) \tilde{\sigma}_1^l \left(\prod_{j=l+1}^{N} \mathrm{e}^{\frac{z}{2}\tilde{\sigma}_2^j} \right) \\ (S_z \otimes S_z \ldots^{N)} \otimes S_z)(\Delta^{(N)}(\tilde{J}_2)) &= \sum_{l=1}^{N} \tilde{\sigma}_2^l \\ (S_z \otimes S_z \ldots^{N)} \otimes S_z)(\Delta^{(N)}(\tilde{J}_3)) &= \sum_{l=1}^{N} \left(\prod_{i=1}^{l-1} \mathrm{e}^{-\frac{z}{2}\tilde{\sigma}_2^i} \right) \tilde{\sigma}_3^l \left(\prod_{j=l+1}^{N} \mathrm{e}^{\frac{z}{2}\tilde{\sigma}_2^j} \right). \end{aligned} \tag{7.7}$$

If we consider now the Hamiltonian function (3.13), its Nth-order coproduct leads, through the usual method, to the following deformation of the clasical Gaudin XYZ system (7.4):

$$
\begin{aligned}
H_z^{(N)}(\tilde{\boldsymbol{\sigma}}) = \sum_{l=1}^{N} &\{(\tilde{\sigma}_2^l)^2 + \mathrm{e}^{2z\beta_l}(\kappa_2(\tilde{\sigma}_1^l)^2 + \kappa_1(\tilde{\sigma}_3^l)^2)\} \\
&+2\sum_{i<j}^{N}\{\tilde{\sigma}_2^i\tilde{\sigma}_2^j + \mathrm{e}^{z\alpha_{ij}}(\kappa_2\tilde{\sigma}_1^i\tilde{\sigma}_1^j + \kappa_1\tilde{\sigma}_3^i\tilde{\sigma}_3^j)\}
\end{aligned}
\tag{7.8}
$$

where the β, α functions depend on $\tilde{\sigma}_2$ as follows:

$$
\begin{aligned}
\beta_i &= -\tfrac{1}{2}\left(\sum_{j=1}^{i-1}\tilde{\sigma}_2^j\right) + \tfrac{1}{2}\left(\sum_{k=i+1}^{N}\tilde{\sigma}_2^k\right) \\
\alpha_{ij} &= \beta_i + \beta_j = -\tfrac{1}{2}(\tilde{\sigma}_2^i - \tilde{\sigma}_2^j) - \sum_{l=1}^{i-1}\tilde{\sigma}_2^l + \sum_{k=j+1}^{N}\tilde{\sigma}_2^k .
\end{aligned}
\tag{7.9}
$$

This Hamiltonian corresponds to a sort of XYZ Gaudin magnet with variable range anisotropy given by the α_{ij} functions. In the limit $z \to 0$ we recover the non-deformed XYZ system (7.4). Note that the commutativity among the $\tilde{\sigma}_i^l$ allows such a compact final expression, that will certainly contain additional terms in the quantum mechanical case. The complete integrability of such a Hamiltonian is ensured by the mth deformed coproducts ($m \leqslant N$) of C_z (2.8) in the S_z representation. A closed expression for them is not difficult to find by recalling formula (6.14):

$$
C_z^{(m)}(\tilde{\boldsymbol{\sigma}}) = \sum_{l=1}^{m}\mathrm{e}^{2z\beta_i}C_z^i + 2\sum_{i<j}^{m}\mathrm{e}^{z\alpha_{ij}}\left\{\frac{\sinh(\frac{z}{2}\tilde{\sigma}_2^i)}{z/2}\frac{\sinh(\frac{z}{2}\tilde{\sigma}_2^j)}{z/2} - \tilde{\sigma}_1^i\tilde{\sigma}_1^j - \tilde{\sigma}_3^i\tilde{\sigma}_3^j\right\}
\tag{7.10}
$$

where C_z^i are the corresponding deformed Casimir functions on each lattice site. As usual, the Nth Casimir can be considered as the Hamiltonian. In that case, any of the coproducts (7.7) can be used to complete the integrals of the motion.

7.2.1. The zero representation. We insist now on the fact that the $\tilde{\sigma}_i$ coordinates are deformed ones. However, realizations in terms of the non-deformed variables σ_j are available. In particular, let us consider the (deformed) Poisson realization U_z

$$
U_z(\tilde{J}_2) = \sigma_2 \qquad U_z(\tilde{J}_1) = \frac{\sinh(\frac{z}{2}\sigma_2)}{\sigma_2 z/2}\sigma_1 \qquad U_z(\tilde{J}_3) = \frac{\sinh(\frac{z}{2}\sigma_2)}{\sigma_2 z/2}\sigma_3.
\tag{7.11}
$$

The functions defined by (7.11) close an $U_z so(2,1)$ under the Poisson bracket (7.1) and provided that the classical coordinates are defined on the $c = 0$ cone $\sigma_2^2 - \sigma_1^2 - \sigma_3^2 = 0$. In this case, the previous construction leads to the following Hamiltonian:

$$
\begin{aligned}
H_z^{(N)}(\boldsymbol{\sigma}) = \sum_{l=1}^{N}&\left\{(\sigma_2^l)^2 + \mathrm{e}^{2z\beta_i}\left(\frac{\sinh(\frac{z}{2}\sigma_2^l)}{\sigma_2^l z/2}\right)^2(\kappa_2(\sigma_1^l)^2 + \kappa_1(\sigma_3^l)^2)\right\} \\
&+2\sum_{i<j}^{N}\left\{\sigma_2^i\sigma_2^j + \mathrm{e}^{z\alpha_{ij}}\frac{\sinh(\frac{z}{2}\sigma_2^i)}{\sigma_2^i z/2}\frac{\sinh(\frac{z}{2}\sigma_2^j)}{\sigma_2^j z/2}(\kappa_2\sigma_1^i\sigma_1^j + \kappa_1\sigma_3^i\sigma_3^j)\right\}.
\end{aligned}
\tag{7.12}
$$

The constants of motion are easily computed and read (in our representation $C_z = 0$):

$$
C_z^{(m)}(\boldsymbol{\sigma}) = 2\sum_{i<j}^{m}\mathrm{e}^{z\alpha_{ij}}\frac{\sinh(\frac{z}{2}\sigma_2^i)}{\sigma_2^i z/2}\frac{\sinh(\frac{z}{2}\sigma_2^j)}{\sigma_2^j z/2}\{\sigma_2^i\sigma_2^j - \sigma_1^i\sigma_1^j - \sigma_3^i\sigma_3^j\}.
\tag{7.13}
$$

In this case, $C_z^{(m)}$ are hyperbolic Gaudin Hamiltonians with variable range exchange. An analysis of long-range Hamiltonians and some examples of variable range interacting systems can be found in [36] and [25], respectively.

8. Concluding remarks

Summarizing, we have demonstrated that any algebra A endowed with a coassociative coproduct Δ (either deformed or not) can be seen as the abstract object that, after choosing a given representation, gives rise in a direct and systematic way to a wide class of N-dimensional integrable systems (with N finite but arbitrary). Within this class of systems, the original coalgebra is not only a set of symmetries, but the algebraic object that generates explicitly the Hamiltonian and the constants of motion. Moreover, the theory can be used to generate both classical and quantum systems by choosing, respectively, either a Poisson or an operatorial realization of A.

The universality of the coalgebra-induced construction that we have presented in this paper suggests a number of further investigations in different contexts. From a general point of view we would like to mention the unsolved question concerning the existence of a Lax formulation for this scheme and its connection with the integrability properties of the known quantum algebra invariant Hamiltonians. On the other hand, a symmetry method in order to decide whether a known system is coalgebra invariant or not would evidently be helpful. In this sense, the long-range interacting nature of our construction is worth emphasizing, although not essential (we recall that known quantum algebra invariant systems usually contain only nearest neighbour interactions).

As a consequence arising at a purely 'classical' level, phase-space realizations D of Lie algebras become relevant tools in order to construct new examples. If such a realization exists in terms of only one pair of canonical coordinates, complete integrability is ensured. However, for Lie algebras with rank greater than one, both the existence of various Casimir functions and the possibility of having D realizations depending on more than one canonical pair have to be taken into account in order to analyse the complete integrability of the system.

The explicit solutions for the examples presented here also deserve further investigations. Known results concern the $so(2,1)$ Calogero system defined through the Casimirs (6.3), that was already solved in [20]. The $N = 2$ deformed motion has also been shown to be solvable (and it includes a deformed period) in [37]. For arbitrary N, the quantum deformation can be seen as a displacement from the geodesic motion (on a proper manifold) that characterizes the non-deformed system. All these results concerning the canonical realization should be completed and translated into the behaviour of the Gaudin systems defined through the angular momentum Poisson bracket.

Finally, we think that these results provide a strong physical motivation for Hopf algebra deformations, since they could now be systematically used to generate new integrable systems (we recall that the $(1+1)$ Poincaré example shows that such deformed systems can be interesting even when their non-deformed counterparts are associated with trivial dynamics). It is known that the number of Hopf algebra deformations for a given $U(g)$ is not arbitrary; in fact, their classification is intimately linked to the notion of Lie bialgebra and, for some low-dimensional cases, complete (and constructive) classifications of quantum deformations have recently been obtained [15]. Therefore, a coalgebra invariant Hamiltonian constructed from a given g can be 'integrably' deformed in a finite number of ways that, at least in some cases, can be explicitly obtained and will certainly provide a better understanding of the physical relevance of coalgebra symmetries.

Acknowledgments

AB has been partially supported by DGICYT (project PB94-1115) from the Ministerio de Educación y Ciencia de España and by Junta de Castilla y León (projects CO1/396 and CO2/297) and acknowledges the hospitality during his stays in Rome; OR has been partially supported by INFN, Sezione di Roma, and is grateful to the Physics Department of Burgos University for its hospitality. The authors would like to thank Francisco Herranz for a careful reading of the manuscript.

Appendix A

The equivalence between definition (4.5) and the usual ones (4.3), (4.4) is obvious for the $N = 3$ case, being expressed in terms of the coassociativity condition (4.1). The case $N = 4$ is also easy to check by direct computation. Therefore, we shall prove the general case by induction, by taking into account that, for a generic $N + 1$, we have to prove that any value of $m = 1, \ldots, N$ in the definition (4.5) leads to (4.3), (4.4).

We shall assume that

$$\Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \circ \Delta^{(2)} \qquad \forall m = 1, \ldots, N-1 \tag{A.1}$$

holds, and we have to prove that

$$\Delta^{(N+1)} := (\Delta^{(k)} \otimes \Delta^{(N-k+1)}) \circ \Delta^{(2)} \qquad \forall k = 1, \ldots, N. \tag{A.2}$$

If we denote $\mathrm{id}^{(r)} \equiv \mathrm{id}\otimes\mathrm{id} \ldots^{r)} \otimes\mathrm{id}$, from (4.4) we can compute $\Delta^{(N+1)}$ in the following way:

$$\begin{aligned}
\Delta^{(N+1)} &= (\Delta^{(2)} \otimes \mathrm{id}^{(N-1)}) \circ \Delta^{(N)} \\
&= (\Delta^{(2)} \otimes \mathrm{id}^{(N-1)}) \circ (\Delta^{(m)} \otimes \Delta^{(N-m)}) \circ \Delta^{(2)} \\
&= (((\Delta^{(2)} \otimes \mathrm{id}^{(m-1)}) \circ \Delta^{(m)}) \otimes \Delta^{(N-m)}) \circ \Delta^{(2)} \\
&= (\Delta^{(m+1)} \otimes \Delta^{(N-m)}) \circ \Delta^{(2)}
\end{aligned} \tag{A.3}$$

where we can choose $\forall m = 1, \ldots, N-1$. Therefore, the validity of (A.2) is proven for $k = 2, \ldots, N$.

The only relation which remains to be proven is the case $k = 1$, that reads

$$\Delta^{(N+1)} := (\mathrm{id} \otimes \Delta^{(N)}) \circ \Delta^{(2)}. \tag{A.4}$$

In this case we can compute its equivalence with respect to the known recurrence (4.4) as follows:

$$\begin{aligned}
\Delta^{(N+1)} &= (\Delta^{(2)} \otimes \mathrm{id}^{(N-1)}) \circ \Delta^{(N)} \\
&= (\Delta^{(2)} \otimes \mathrm{id} \otimes \mathrm{id}^{(N-2)}) \circ \Delta^{(N)} \\
&= (\Delta^{(2)} \otimes \mathrm{id} \otimes \mathrm{id}^{(N-2)}) \circ (\Delta^{(2)} \otimes \Delta^{(N-2)}) \circ \Delta^{(2)} \\
&= (((\Delta^{(2)} \otimes \mathrm{id}) \circ \Delta^{(2)}) \otimes \Delta^{(N-2)}) \circ \Delta^{(2)} \\
&= (((\mathrm{id} \otimes \Delta^{(2)}) \circ \Delta^{(2)}) \otimes \Delta^{(N-2)}) \circ \Delta^{(2)} \\
&= (\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(N-2)}) \circ (\Delta^{(2)} \otimes \Delta^{(N-2)}) \circ \Delta^{(2)} \\
&= (\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(N-2)}) \circ \Delta^{(N)} \\
&= (\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(N-2)}) \circ (\mathrm{id} \otimes \Delta^{(N-1)}) \circ \Delta^{(2)} \\
&= (\mathrm{id} \otimes ((\Delta^{(2)} \otimes \mathrm{id}^{(N-2)}) \circ \Delta^{(N-1)})) \circ \Delta^{(2)} \\
&= (\mathrm{id} \otimes \Delta^{(N)}) \circ \Delta^{(2)}.
\end{aligned} \tag{A.5}$$

Finally, note that the equivalence between the ordinary definitions (4.3) and (4.4) is obtained as a byproduct from this derivation by considering the $k = N$ case.

Appendix B

The aim of this appendix is to prove the homomorphism condition (4.8) that we shall split into the commutator and Poisson cases, respectively. As usual, the $N = 2$ case is part of the definition of a (Poisson) Hopf algebra, and we shall proceed by induction.

B.1. $\Delta^{(N)}$ as a homomorphism

Let us consider the algebra A endowed with an associative product $(\cdot)$ that we shall now explicitly write. We know that, by definition, the coproduct $\Delta^{(2)}$ is a homomorphism between A and $A \otimes A$:

$$\Delta^{(2)}(X \cdot Y) = \Delta^{(2)}(X) \cdot \Delta^{(2)}(Y) \qquad \forall X, Y \in A. \tag{B.1}$$

If we assume that $\Delta^{(N-1)}$ is a homomorphism, by using Sweedler's notation,

$$\Delta^{(2)}(X) = \sum_\alpha X_{1\alpha} \otimes X_{2\alpha} \qquad \Delta^{(2)}(Y) = \sum_\beta Y_{1\beta} \otimes Y_{2\beta} \tag{B.2}$$

and by recalling the definition of $\Delta^{(N)}$ in terms of $\Delta^{(N-1)}$ and $\Delta^{(2)}$, we have that

$$\begin{aligned}
\Delta^{(N)}(X) \cdot \Delta^{(N)}(Y) &= ((\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)}(X)) \cdot ((\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)}(Y)) \\
&= \sum_{\alpha,\beta} \left(\Delta^{(N-1)}(X_{1\alpha}) \otimes X_{2\alpha}\right) \cdot \left(\Delta^{(N-1)}(Y_{1\beta}) \otimes Y_{2\beta}\right) \\
&= \sum_{\alpha,\beta} \Delta^{(N-1)}(X_{1\alpha} \cdot Y_{1\beta}) \otimes X_{2\alpha} \cdot Y_{2\beta} \\
&= (\Delta^{(N-1)} \otimes \mathrm{id}) \left(\sum_{\alpha,\beta} X_{1\alpha} \cdot Y_{1\beta} \otimes X_{2\alpha} \cdot Y_{2\beta} \right) \\
&= (\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)}(X \cdot Y) = \Delta^{(N)}(X \cdot Y).
\end{aligned} \tag{B.3}$$

This result holds for $(\cdot)$ being either a commutative or a non-commutative product. In the latter case, the homomorphism condition for the commutator $[X, Y] := X \cdot Y - Y \cdot X$ is immediately deduced from this result.

B.2. $\Delta^{(N)}$ as a Poisson map

Let us assume that $(A, \Delta^{(2)})$ is a Poisson–Hopf algebra and that the $(N-1)$th coproduct fulfils

$$\{\Delta^{(N-1)}(X), \Delta^{(N-1)}(Y)\}_{A\otimes A\otimes \ldots^{N-1)}\otimes A} = \Delta^{(N-1)}(\{X, Y\}_A) \qquad \forall X, Y \in A. \tag{B.4}$$

(Hereafter we shall supress the subscripts that label the space where the Poisson bracket is defined.) From (4.5) we can write

$$\{\Delta^{(N)}(X), \Delta^{(N)}(Y)\} = \{(\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)}(X), (\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)}(Y)\}. \tag{B.5}$$

With the aid of (B.2) we can compute it explicitly:

$$\{\Delta^{(N)}(X), \Delta^{(N)}(Y)\} = \left\{ (\Delta^{(N-1)} \otimes \mathrm{id}) \left(\sum_\alpha X_{1\alpha} \otimes X_{2\alpha} \right), (\Delta^{(N-1)} \otimes \mathrm{id}) \left(\sum_\beta Y_{1\beta} \otimes Y_{2\beta} \right) \right\}$$

$$
\begin{aligned}
&= \sum_{\alpha,\beta} \{\Delta^{(N-1)}(X_{1\alpha}) \otimes X_{2\alpha}, \Delta^{(N-1)}(Y_{1\beta}) \otimes Y_{2\beta}\} \\
&= \sum_{\alpha,\beta} (\{\Delta^{(N-1)}(X_{1\alpha}), \Delta^{(N-1)}(Y_{1\beta})\} \otimes X_{2\alpha} \cdot Y_{2\beta} \\
&\quad + (\Delta^{(N-1)}(X_{1\alpha}) \cdot \Delta^{(N-1)}(Y_{1\beta})) \otimes \{X_{2\alpha}, Y_{2\beta}\}) \\
&= \sum_{\alpha,\beta} (\Delta^{(N-1)}(\{X_{1\alpha}, Y_{1\beta}\}) \otimes (X_{2\alpha} \cdot Y_{2\beta}) + \Delta^{(N-1)}(X_{1\alpha} \cdot Y_{1\beta}) \otimes \{X_{2\alpha}, Y_{2\beta}\}) \\
&= \sum_{\alpha,\beta} (\Delta^{(N-1)} \otimes \mathrm{id})(\{X_{1\alpha}, Y_{1\beta}\} \otimes (X_{2\alpha} \cdot Y_{2\beta}) + (X_{1\alpha} \cdot Y_{1\beta}) \otimes \{X_{2\alpha}, Y_{2\beta}\}) \\
&= \sum_{\alpha,\beta} (\Delta^{(N-1)} \otimes \mathrm{id})(\{X_{1\alpha} \otimes X_{2\alpha}, Y_{1\beta} \otimes Y_{2\beta}\}) \\
&= (\Delta^{(N-1)} \otimes \mathrm{id})\left(\left\{ \sum_{\alpha} X_{1\alpha} \otimes X_{2\alpha}, \sum_{\beta} Y_{1\beta} \otimes Y_{2\beta} \right\}\right) \\
&= ((\Delta^{(N-1)} \otimes \mathrm{id}) \circ \Delta^{(2)})(\{X, Y\}) = \Delta^{(N)}(\{X, Y\}).
\end{aligned} \tag{B.6}
$$

Throughout this computation we have used the Poisson-map condition (B.4) for $\Delta^{(N-1)}$ and the homomorphism condition for the (now commutative) $(\cdot)$ product in the Poisson algebra. Note that the proof (b) of the commutative (Poisson) case is more involved. In this classical mechanical context we have to impose the compatibility of Δ with respect to *two* independent products: the (commutative) ‘pointwise’ one $(\cdot)$ and the Poisson bracket $\{,\}$. In contrast, in the ‘quantum-mechanical’ case the latter is replaced by the commutator, which is constructed in terms of the former (now non-commutative).

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